ON THE OPTIMAL CONVERGENCE RATE OF A ROBIN-ROBIN DOMAIN DECOMPOSITION METHOD

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ABSTRACT. In this work, we solve a long-standing open problem: Is it true that the convergence rate of the Lions' Robin-Robin nonoverlapping domain decomposition(DD) method can be constant, independent of the mesh size h? We closed this twenty-year old problem with a positive answer. Our theory is also verified by numerical tests.

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1. Introduction

Domain decomposition (DD) methods are important tools for solving partial differential equations, especially by parallel computers. In this paper, we shall study a class of nonoverlapping DD method, which is based on using Robin-Robin boundary conditions as transmission conditions on the subdomain interface. The idea of employing Robin-Robin coupling conditions in DD methods was first proposed by P.L. Lions in [24]. In the past twenty years, there are many works on the analysis and applications of this DD method: Despres [8], Douglas and Huang [12, 13], Deng [6, 7], Du [14], Gander *et al.* [20, 21], Guo and Hou [22], Discacciati [9], Flauraud and Nataf [16], Gander [17, 19], Qin and Xu [26, 27, 28], Discacciati *et al.* [10], Lui [25], and Chen *et al.* [2, 3]. We should say that the list is far from being complete.

By comparison with other DD methods, Lions' DD method has several advantages. The iterative procedure is simple and much more highly parallel than others. Because it employs Robin conditions, the method is specially suitable for solving Helmholtz and time-harmonic Maxwell equations. There exists a lot of works in this direction, cf. [8, 1, 21, 11] for details.

Lions' Robin-Robin DD method was proposed in 1990 [24], see Definition 1.1 below (without Step 5). The convergence (without any rate) is shown in [24, 29]. Later, the convergence was improved to a geometric convergence [13, 12, 22], i.e, a rate of $1-O(h)$. It was first pointed out by Gander, Halpern and Nataf in [20] that the optimal choice of relaxation parameter is $\gamma = O(h^{-1/2})$ and the convergence

rate $1-O(\sqrt{h})$ could be achieved. Recently, Xu and Qin [30] give a rigorous analysis on this result and shows that the rate is asymptotically sharp. However, without enough knowledge on the method, the two parameters γ_1 and γ_2 in Lions' DD method are set equal, see Definition 1.1 below, by researchers in above references. Thus, the rate of $1-O(\sqrt{h})$ is generally believed optimal for the Lions' DD method.

This paper answers this long-standing open problem: Is it possible to achieve a rate of $1 - C$ for some some constant $C > 0$ independent of the mesh size h? We give a positive answer. Yes, the constant rate of convergence is achieved by well-choosing three parameters in the Robin-Robin DD method, γ_1, γ_2 and θ, in Definition 1.1. Roughly speaking, the optimal choices are

$$\gamma_1 = O(1), \ \gamma_2 = O(h^{-1}), \quad \text{and} \quad \theta = \frac{2t-1}{2t+1},$$

where $t \approx 1$ is the ratio of spectral radii of two Dirichlet-Neumann operators on two subdomains. It is shown in this paper, by three types of analysis, that the error reduction rate of the DD method is optimal, $1 - C$.

Next, we introduce the Robin-Robin DD method through a simple model problem. We solve the following model problem in 2D, which is decomposed into two subproblems (cf. Figure 1):

$$\left\{ \begin{aligned} -\Delta u &= f && \text{in } \Omega_1, \\ u &= 0 && \text{on } \partial\Omega \cap \partial\Omega_1, \\ u - w &= \frac{\partial u}{\partial \mathbf{n}} - \frac{\partial w}{\partial \mathbf{n}} = 0 && \text{on } \Gamma, \\ -\Delta w &= f && \text{in } \Omega_2, \\ u &= 0 && \text{on } \partial\Omega \cap \partial\Omega_2, \end{aligned} \right. \tag{1.1}$$

where Γ is an interface separating Ω_1 and Ω_2, and $\mathbf{n}$ is an outward normal vector of Ω_1 at Γ. The DD method can be applied to general elliptic PDEs, general domains and multiple subdomains, cf. [6, 29].

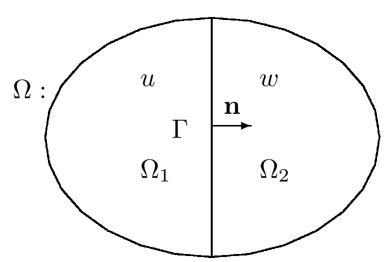

FIGURE 1. A domain is decomposed into two subdomains.

The Dirichlet and Neumann interface conditions on Γ in (1.1) are combined into two Robin interface conditions:

$$\gamma_1 u + \frac{\partial u}{\partial \mathbf{n}} = \gamma_1 w + \frac{\partial w}{\partial \mathbf{n}} = g_1 \qquad \text{on } \Gamma, \tag{1.2}$$

$$\gamma_2 u - \frac{\partial u}{\partial \mathbf{n}} = \gamma_2 w - \frac{\partial w}{\partial \mathbf{n}} = g_2 \qquad \text{on } \Gamma. \tag{1.3}$$

Here we allow γ_1, γ_2 to be any positive constants. For example, when γ_1 is arbitrarily close to zero and γ_2 is close to infinity (but the linear systems would become near singular), the method would be reduced to the Dirichlet-Neumann DD method. The past researchers all set $\gamma_1 = \gamma_2 = \gamma$ in the Robin interface conditions, i.e., the two parameters are simultaneously large or small. By selecting two parameters correctly, using the original Lions' DD method, this Robin-Robin domain decomposition method should be better than all existing Dirichlet-Neumann, Neumann-Neumann and Robin-Robin domain decomposition methods.

Let $V_i = H_0^1(\Omega)|_{\Omega_i}$. Later, V_i also denotes the restriction of the finite element space of grid size h on the two subdomains Ω_i. By (1.2), we do an integration by parts on Ω_1 to get

$$\begin{aligned}\int_\Gamma g_1 v ds &= \int_\Gamma (\frac{\partial u}{\partial \mathbf{n}} + \gamma_1 u) v ds = \int_{\Omega_1} (\nabla u \cdot \nabla v + \Delta u v) d\mathbf{x} + \gamma_1 \int_\Gamma uvds \\ &= \int_{\Omega_1} (\nabla u \cdot \nabla v - fv) d\mathbf{x} + \gamma_1 \int_\Gamma uvds.\end{aligned}$$

Thus

$$a_1(u, v) + \gamma_1 \langle u, v\rangle = (f, v)_{\Omega_1} + \langle g_1, v\rangle \quad \forall v \in V_1,$$

where

$$\begin{aligned}a_i(u, v) &= \int_{\Omega_i} \nabla u \cdot \nabla v d\mathbf{x}, \quad i = 1, 2, \\ (f, v)_{\Omega_i} &= \int_{\Omega_i} fv d\mathbf{x}, \quad i = 1, 2, \\ \langle u, v\rangle &= \int_\Gamma uvds.\end{aligned}$$

Similarly, by (1.3) and an integration by parts on Ω_2, it follows (noting that $\mathbf{n}$ is an inward normal vector to Ω_2) that

$$\begin{aligned}\int_\Gamma g_2 v ds &= \int_\Gamma (\gamma_2 w - \frac{\partial w}{\partial \mathbf{n}}) v ds = \int_{\Omega_2} (\nabla w \cdot \nabla v + \Delta w v) d\mathbf{x} + \gamma_2 \int_\Gamma wvds \\ v &= \int_{\Omega_2} (\nabla w \cdot \nabla v - fv) d\mathbf{x} + \gamma_2 \int_\Gamma wvds.\end{aligned}$$

This way, we get the second variational problem on Ω_2:

$$a_2(w, v) + \gamma_2 \langle w, v\rangle = (f, v)_{\Omega_2} + \langle g_2, v\rangle \quad \forall v \in V_2.$$

Definition 1.1. *(The Robin-Robin DD method.)* *Given $g_1^0 (= 0)$ on Γ, a serial version domain decomposition iteration consists the following five steps ($m = 0, 1, \dots$):*

(1) *Solve on Ω_1 for u^m:*

$$a_1(u^m, v) + \gamma_1\langle u^m, v\rangle = (f, v)_{\Omega_1} + \langle g_1^m, v\rangle \quad \forall v \in V_1. \tag{1.4}$$

(2) *Update the interface condition on Γ:*

$$g_2^m = -g_1^m + \big(\gamma_2 + \gamma_1\big)u^m. \tag{1.5}$$

(3) *Solve on Ω_2 for w^m:*

$$a_2(w^m, v) + \gamma_2\langle w^m, v\rangle = (f, v)_{\Omega_2} + \langle g_2^m, v\rangle \quad \forall v \in V_2. \tag{1.6}$$

(4) *Update the other interface condition on Γ:*

$$\tilde{g}_1^m = -g_2^m + \big(\gamma_1 + \gamma_2\big)w^m. \tag{1.7}$$

(5) *Get the next iterate by a relaxation:*

$$g_1^{m+1} = \theta g_1^m + (1-\theta)\tilde{g}_1^m. \tag{1.8}$$

The rest of paper is organized as follows. In section 2, we shall show that although the Robin-Robin DD method cannot achieve the geometrical convergence rate at the continuous PDE level, but it does at the discrete level. In section 3, we shall give an explicit convergence rate of the DD method on uniform meshes. In section 4, we shall extend our method to more general quasi-uniform meshes. Using the Dirichlet-to-Neumann operator, we shall prove that the Robin-Robin DD method is optimal. Finally, in the last section, we shall present some numerical results to support our theory. It is seen from our numerical implementation that this DD method is better than Dirichlet-Neumann DD method and one-parameter Robin-Robin DD method.

2. A Von Neumann analysis

In this section, through a simple model problem, we shall show that for the new DD method it is not possible to get the geometrical convergence rate strictly less than one at the continuous level, but it is possible at the discrete level.

Let us assume that $\Omega_1 = [-\pi, 0] \times [0, \pi]$ and $\Omega_2 = [0, \pi] \times [0, \pi]$, and it is enough for us to assume that $f \equiv 0$ so that the true solutions of Equation (1.1) vanishes. Now if $g_1 = \hat{g}_1 \sin ky$ on Γ, from Equations (1.1) and (1.2), the solution on Ω_1 is

$$u = \hat{u}\sinh(k(x+1))\sin ky, \quad \text{where} \quad \hat{u} = \frac{\hat{g}_1}{\gamma_1 \sinh k + k\cosh k}.$$

If $g_2 = \hat{g}_2 \sin ky$ on Γ, from Equations (1.1) and (1.3), the solution on Ω_2 is

$$w = \hat{w}\sinh(k(x-1))\sin ky, \quad \text{where} \quad \hat{w} = -\frac{\hat{g}_2}{\gamma_2 \sinh k + k\cosh k}.$$

By Definition 1.1, if the initial error is $g_1^m = \hat{g}_1^m \sin ky$ on Γ, then

$$g_2^m = \hat{g}_2^m \sin ky, \quad \text{where} \quad \hat{g}_2^m = \hat{g}_1^m\left(\frac{\gamma_2+\gamma_1}{\gamma_1 + k\coth k} - 1\right).$$

Then, by (1.6) and (1.7),

$$\tilde{g}_1^m = \hat{\tilde{g}}_1^m \sin ky, \quad \text{where} \quad \hat{\tilde{g}}_1^m = \hat{g}_2^m \left(\frac{\gamma_2 + \gamma_1}{\gamma_2 + k \coth k} - 1 \right).$$

Finally, after the relaxation step (1.8),

$$g_1^{m+1} = \hat{g}_1^{m+1} \sin ky,$$

where

$$\hat{g}_1^{m+1} = \theta \hat{g}_1^m + (1-\theta)\hat{\tilde{g}}_1^m = \rho \hat{g}_1^m,$$

and the factor

$$\rho = \theta + (1-\theta) \left(\frac{\gamma_2 + \gamma_1}{\gamma_2 + k \coth k} - 1 \right) \left(\frac{\gamma_2 + \gamma_1}{\gamma_1 + k \coth k} - 1 \right). \tag{2.1}$$

Now for the fixed parameters $0 < \theta < 1$, $\gamma_1 > 0$ and $\gamma_2 > 0$, if k tends to infinity, then

$$\rho \approx 1 - 2(1-\theta)\frac{\gamma_1 + \gamma_2}{k} \to 0. \tag{2.2}$$

Therefore in the continuous case, it is impossible to get the convergence rate independent of the frequency (or the wave number) k. On the other hand, if k is bounded by $1 \le k \le K$, we may obtain the convergence rate ρ , which is independent of k (but dependent on K), through choosing the three parameters γ_1, γ_2 and θ.

Lemma 2.1. *If a and b are two non-negative constants, then the function*

$$\rho(\theta) = \max\{|\theta - (1-\theta)a|, |\theta - (1-\theta)b|\} \tag{2.3}$$

attains the minimum value $\frac{|b-a|}{2+a+b}$ at $\theta_0 = \frac{a+b}{2+a+b}$.

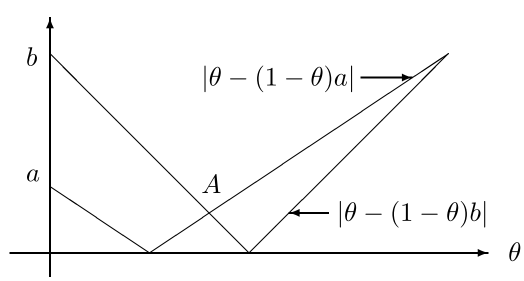

Figure 2. Graphs of $|\theta - (1-\theta)a|$ and $|\theta - (1-\theta)b|$, $b > a \ge 0$.

Proof. Without loss of generality, we assume $b \ge a$. Both terms in (2.3) are piecewise linear functions. We plot them in Figure 2. The minimal value is attained at the point A, where two lines intersect:

$$\theta - (1-\theta)a + \theta - (1-\theta)b = 0.$$

That is $\theta = \theta_0 = \frac{a+b}{2+a+b}$, and

$$|\theta_0 - (1-\theta_0)a| = |\theta_0 - (1-\theta_0)b| = \frac{a+b}{2+a+b}.$$

So we get the lemma. □

Lemma 2.2. *For any $z \geq 0$, the function*

$$\omega(z) = \frac{\gamma_2 - z}{\gamma_2 + z} \cdot \frac{z - \gamma_1}{z + \gamma_1} \tag{2.4}$$

attains the maximum value at $z_0 = \sqrt{\gamma_1 \gamma_2}$:

$$\max_{z>0} \omega(z) = \frac{(\eta - 1)^2}{(\eta + 1)^2}, \quad \text{where} \quad \eta = \sqrt{\frac{\gamma_2}{\gamma_1}}. \tag{2.5}$$

Proof. The derivative of $\omega(z)$ is

$$\omega'(z) = \frac{2(\gamma_1 + \gamma_2)(\gamma_1\gamma_2 - z^2)}{(z+\gamma_1)^2 (z+\gamma_2)^2}. \tag{2.6}$$

So $\omega(z)$ monotonically increases when $z < z_0$ and monotonically decreases when $z > z_0$. In particular, then the minimum value of $\omega(z)$ on an interval $[z_1, z_2]$ is attained at one of the end points:

$$\min_{z \in [z_1, z_2]} \omega(z) = \min\{\omega(z_1), \omega(z_2)\}. \tag{2.7}$$

By (2.6), $\omega(z)$ attains the only global maximum value at $z_0 = \sqrt{\gamma_1\gamma_2}$:

$$\omega(z_0) = \frac{\gamma_2 - \sqrt{\gamma_1\gamma_2}}{\sqrt{\gamma_1\gamma_2} + \gamma_2} \frac{\sqrt{\gamma_1\gamma_2} - \gamma_1}{\sqrt{\gamma_1\gamma_2} + \gamma_1} = \frac{(\eta-1)^2}{(\eta+1)^2}.$$

The lemma is proved. □

From Equation (2.1),

$$\rho = \theta - (1-\theta)\frac{\gamma_2 - k\coth k}{\gamma_2 + k \coth k} \cdot \frac{k \coth k - \gamma_1}{\gamma_1 + k \coth k} = \theta - (1-\theta)\,\omega(k\coth k).$$

If γ_1 and γ_2 are chosen such that $\gamma_1 < \coth 1$ and $\gamma_2 > K \coth K$, then $\omega(k \coth k) > 0$. By Lemma 2.2,

$$|\rho| \leq \max\{|\theta|, |\theta - (1-\theta)\omega(z_0)|\}.$$

Applying Lemma 2.1, we may select

$$\theta_0 = \frac{\omega(z_0)}{2 + \omega(z_0)} \quad \Rightarrow \quad |\rho| \leq |\theta_0| < \frac{1}{3}. \tag{2.8}$$

Remark 2.1. *If $\eta > \frac{\sqrt{2}+1}{\sqrt{2}-1}$, we may just set $\theta = \frac{1}{3}$, and $|\rho|$ is also less than $\frac{1}{3}$. Moreover, this bound can be improved further if we carefully estimate the minimum value of $\omega(z)$.*

Remark 2.2. *The constrain $\gamma_1 < \coth 1$ can be relaxed. Actually, if $\gamma_1 > \coth 1$, then*

$$|\rho| \leq \max\left\{|\theta + (1-\theta)\zeta|\,, |\theta - (1-\theta)\omega(z_0)|\right\},$$

where $\zeta = \frac{\gamma_1 - \coth 1}{\gamma_1 + \coth 1}$. *Then we set* $\theta = 0$ *if* $\omega(z_0) \le \zeta$ *and set* $\theta = \frac{\omega(z_0)-\zeta}{2+\omega(z_0)-\zeta}$ *if* $\omega(z_0) > \zeta$*, and*

$$|\rho| \le \begin{cases} \zeta, & \text{if } \omega(z_0) \le \zeta, \\ \frac{\omega(z_0)+\zeta}{2+\omega(z_0)-\zeta}, & \text{if } \omega(z_0) > \zeta. \end{cases}$$

Note that $\omega(z_0) < 1$ *and* $\frac{\omega(z_0)+\zeta}{2+\omega(z_0)-\zeta} \le \frac{1+\zeta}{3-\zeta}$ *which is also independent of* K*.*

The Von Neumann analysis shows that the Robin-Robin DD does have a constant rate of convergence, independent of the frequency number k or K. But the selection of the two parameters depends on K. The limit case indicates that the method deteriorates to, i.e., $\gamma_2 = \infty$, a Robin-Dirichlet DD method.

3. Convergence on uniform grids

In this section, we analyze the Robin-Robin DD method on uniform grids. In this case, we give explicit eigenvalues of the iterative matrix, and show the optimal rate of convergence.

We post a uniform grid of size $h = 1/(2n)$ on the domain $\Omega = [0,1]^2$, the unit square, shown in Figure 4. Then, we subdivide the domain into two, as shown in Figure 3. We give two numberings of nodal values of the C^0-P_1 finite element functions. One numbering is on the interface Γ. The other one is within each subdomain, Ω_1 and Ω_2. When numbering the nodes in Ω_2, we go from right to left so that the nodal index is symmetric to that on the left domain Ω_1.

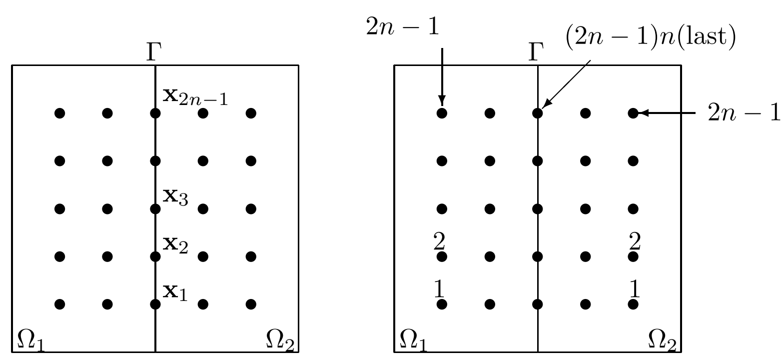

Figure 3. Nodal basis numberings, on Γ and on $\Omega_1 \cap \Omega_2$.

Let M_Γ and A_Γ be two tridiagonal $(2n-1)\times(2n-1)$ matrices:

$$M_\Gamma = \frac{h}{6}\begin{pmatrix} 4 & 1 & & \\ 1 & 4 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 4 \end{pmatrix}, \quad A_\Gamma = \frac{1}{2}\begin{pmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{pmatrix}.$$

Here M_Γ is just the mass matrix of the inner product $\langle\cdot,\cdot\rangle$. Let R_h be the $(2n-1)\times(2n-1)n$ matrix representing a restriction operator on Γ:

$$R_h = (0_{2n-1}, \cdots, 0_{2n-1}, I_{2n-1}). \tag{3.1}$$

The stiffness matrix of the bilinear form $a_1(\cdot,\cdot)$, under nodal basis, (and $a_2(\cdot,\cdot)$ too) is

$$A_h = A_0 - R_h^T A_\Gamma R_h,$$

where the matrix A_0 is the stiffness matrix of size $(2n-1)n$, for the Laplace operator on a $(2n)\times(n+1)$ uniform grid with zero Dirichlet boundary condition. A_0 is same as the matrix of standard five-point finite difference matrix, which has the eigen-decomposition [5, 23]:

$$A_0 = (\Phi_n\otimes\Phi_{2n-1})^T(\Lambda_n\otimes I_{2n-1} + I_n\otimes\Lambda_{2n-1})(\Phi_n\otimes\Phi_{2n-1}), \tag{3.2}$$

where Λ_m denotes an diagonal matrix whose (i,i)-th entry is

$$\lambda_i^{(m)} = 4\sin^2\frac{i\pi}{2(m+1)}, \tag{3.3}$$

and Φ_m denotes an orthogonal matrix defined by

$$\Phi_m = \begin{pmatrix}\phi_1^{(m)} & \cdots & \phi_m^{(m)}\end{pmatrix}, \quad \text{with } \phi_i^{(m)} = \sqrt{\frac{2}{m}}\begin{pmatrix}\sin\frac{i\pi}{m+1}\\ \sin\frac{2i\pi}{m+1}\\ \vdots\\ \sin\frac{mi\pi}{m+1}\end{pmatrix}. \tag{3.4}$$

Here in (3.2), a tensor product matrix $C_{mk\times mk} = A_{m\times m}\otimes B_{k\times k}$ is defined with the (i,j)-th entry

$$C_{ij} = A_{i',j'}B_{i'',j''}, \quad \text{where } \begin{aligned} i &= (i'-1)k + i'',\\ j &= (j'-1)k + j''.\end{aligned}$$

In Definition 1.1, for (1.4), the error $e_u^m = u - u^m$ satisfies the equation:

$$a_1(e_u^m, v) + \gamma_1\langle e_u^m, v\rangle = \langle e_{g_1}^m, v\rangle \quad \forall v\in V_1.$$

Here $e_{g_1}^m = g_1 - g_1^m$ is the error. In the matrix-vector form,

$$E_u^m = (A_h + \gamma_1 R_h^T M_\Gamma R_h)^{-1} R_h^T M_\Gamma E_{g_1}^m.$$

Here E_u^m is the vector representation of e_u^m. Therefore, by (1.6),

$$E_{g_2}^m = \left(-I + (\gamma_2+\gamma_1)R_h(A_h + \gamma_1 R_h^T M_\Gamma R_h)^{-1}R_h^T M_\Gamma\right)E_{g_1}^m. \tag{3.5}$$

Symmetrically, by (1.7) and (1.8),

$$\tilde{E}_{g_1}^m = C_{\gamma_2}E_{g_2}^m. \tag{3.6}$$

Here, for simplicity, we denote the error reduction matrix by

$$C_{\gamma_k} = \left(-I + (\gamma_2+\gamma_1)R_h(A_h + \gamma_k R_h^T M_\Gamma R_h)^{-1}R_h^T M_\Gamma\right), \ k = 1,2. \tag{3.7}$$

Finally, by (1.8), one Robin-Robin DD iteration reduces the initial error $E_{g_1}^m$ to

$$E_{g_1}^{m+1} = [\theta I + (1-\theta)C_{\gamma_2}C_{\gamma_1}]E_{g_1}^m. \tag{3.8}$$

We find the eigenvalue range of this error reduction matrix, via common eigenvectors of all matrices.

Lemma 3.1. *The error reduction matrix* (3.8) *can be diagonalized by* Φ_{2n-1} *defined in* (3.4)*. That is,*

$$\Phi_{2n-1}[\theta I + (1-\theta)C_{\gamma_2}C_{\gamma_1}]\Phi_{2n-1}^T = \mathrm{diag}(\theta + (1-\theta)c_j), \tag{3.9}$$

where in the j*-th diagonal element,*

$$c_j = \frac{\gamma_1 a_j - b_j}{\gamma_1 a_j + b_j} \cdot \frac{\gamma_2 a_j - b_j}{\gamma_2 a_j + b_j}. \tag{3.10}$$

Here in (3.10)*,*

$$a_j = \lambda_{M_\Gamma,j}\tilde{\lambda}_j, \qquad \lambda_{M_\Gamma,j} = h - \frac{h}{6}\lambda_j^{(2n-1)}, \tag{3.11}$$

$$b_j = 1 - \lambda_{A_\Gamma,j}\tilde{\lambda}_j, \qquad \lambda_{A_\Gamma,j} = 1 + \frac{1}{2}\lambda_j^{(2n-1)}, \tag{3.12}$$

where $\lambda_j^{(2n-1)}$ *is defined in* (3.3) *and*

$$\tilde{\lambda}_j = \frac{2}{n+1}\sum_{i=1}^{n} \sin^2\frac{in\pi}{n+1}(\lambda_i^{(n)} + \lambda_j^{(2n-1)})^{-1}. \tag{3.13}$$

Proof. In (3.5), by the Sherman-Morrison-Woodbury formula,

$$\begin{aligned}
&(A_h + \gamma_1 R_h^T M_\Gamma R_h)^{-1} \\
={}& (A_0 + R_h^T(-A_\Gamma + \gamma_1 M_\Gamma)R_h)^{-1} \\
={}& A_0^{-1} - A_0^{-1}R_h^T((-A_\Gamma + \gamma_1 M_\Gamma)^{-1} + R_h A_0^{-1} R_h^T)^{-1} R_h A_0^{-1}.
\end{aligned}$$

Now letting $B_0 = R_h A_0^{-1} R_h^T$, we have

$$R_h(A_h + \gamma_1 R_h^T M_\Gamma R_h)^{-1}R_h^T = B_0 - B_0((-A_\Gamma + \gamma_1 M_\Gamma)^{-1} + B_0)^{-1}B_0.$$

By (3.1) and (3.2), notice that $(\Phi_n \otimes \Phi_{2n-1})R_h^T = \phi_n^{(n)} \otimes \Phi_{2n-1}$, we can compute B_0:

$$\begin{aligned}
B_0 &= (\phi_n^{(n)} \otimes \Phi_{2n-1})^T(\Lambda_n \otimes I_{2n-1} + I_n \otimes \Lambda_{2n-1})(\phi_n^{(n)} \otimes \Phi_{2n-1}) \\
&= \sum_{i=1}^{n}(\phi_{n,i}^{(n)})^2 \Phi_{2n-1}^T(\lambda_i^{(2n-1)} I_{2n-1} + \Lambda_{2n-1})^{-1}\Phi_{2n-1} \\
&= \Phi_{2n-1}^T\left(\sum_{i=1}^{n}(\phi_{n,i}^{(n)})^2(\lambda_i^{(2n-1)} I_{2n-1} + \Lambda_{2n-1})^{-1}\right)\Phi_{2n-1} \\
&= \Phi_{2n-1}^T \tilde{\Lambda}_0 \Phi_{2n-1},
\end{aligned}$$

where $\phi_{n,i}^{(n)}$ is the i-th entry of vector $\phi_n^{(n)}$ defined in (3.4), and $\tilde{\Lambda}_0$ is a diagonal matrix, whose (j,j)-th entry is define in (3.13). The matrices on Γ are diagonalized as $M_\Gamma = \Phi_{2n-1}^T\,\mathrm{diag}(\lambda_{M_\Gamma,j})\Phi_{2n-1}$ and $A_\Gamma = \Phi_{2n-1}^T\,\mathrm{diag}(\lambda_{A_\Gamma,j})\Phi_{2n-1}$, where $\lambda_{M_\Gamma,j}$

and $\lambda_{A_\Gamma,j}$ are defined in (3.11) and (3.12), respectively. Thus combining last two equalities, we get

$$\begin{aligned}
&R_h(A_h+\gamma_1 R_h^T M_\Gamma R_h)^{-1}R_h^T\\
&= B_0 - B_0(\Phi_{2n-1}^T(-\operatorname{diag}(\lambda_{A_\Gamma,j})+\gamma_1\operatorname{diag}(\lambda_{M_\Gamma,j}))^{-1}\Phi_{2n-1}+B_0)^{-1}B_0\\
&= \Phi_{2n-1}^T\left[\tilde\Lambda_0-\tilde\Lambda_0^2\{(-\operatorname{diag}(\lambda_{A_\Gamma,j})+\gamma_1\operatorname{diag}(\lambda_{M_\Gamma,j}))^{-1}+\tilde\Lambda_0\}^{-1}\right]\Phi_{2n-1}\\
&= \Phi_{2n-1}^T\left[\tilde\Lambda_0^{-1}-\operatorname{diag}(\lambda_{A_\Gamma,j})+\gamma_1\operatorname{diag}(\lambda_{M_\Gamma,j})\right]^{-1}\Phi_{2n-1}.
\end{aligned}$$

By (3.7),

$$\begin{aligned}
C_{\gamma_1} &= \Phi_{2n-1}^T(-I+(\gamma_2+\gamma_1)\left[(\tilde\Lambda_0^{-1}-\operatorname{diag}(\lambda_{A_\Gamma,j}))\operatorname{diag}(\lambda_{M_\Gamma,j}^{-1})+\gamma_1 I)\right]^{-1})\Phi_{2n-1}\\
&= \Phi_{2n-1}^T\operatorname{diag}\left(\frac{-1+\gamma_2\lambda_{M_\Gamma,j}\tilde\lambda_j+\lambda_{A_\Gamma,j}\tilde\lambda_j}{1+\gamma_1\lambda_{M_\Gamma,j}\tilde\lambda_j-\lambda_{A_\Gamma,j}\tilde\lambda_j}\right)\Phi_{2n-1}\\
&= \Phi_{2n-1}^T\operatorname{diag}\left(\frac{\gamma_2 a_j-b_j}{\gamma_1 a_j+b_j}\right)\Phi_{2n-1}.
\end{aligned}$$

In the same fashion, it follows that

$$C_{\gamma_2} = \Phi_{2n-1}^T\operatorname{diag}\left(\frac{\gamma_1 a_j-b_j}{\gamma_2 a_j+b_j}\right)\Phi_{2n-1}.$$

Thus (3.9) follows. □

In next lemma, we estimate the eigenvalue c_j in the reduction matrix, (3.10).

Lemma 3.2. *$(3a_j-b_j)$ is monotonically decreasing, i.e., $j=1,\dots,2n-2$,*

$$3a_j-b_j \ge 3a_{j+1}-b_{j+1}. \tag{3.14}$$

Proof. We rewrite the $\tilde\lambda_j$ (in a_j and b_j) in a symmetric form so that each i-term is a decreasing function of j (the original term is not.)

$$\begin{aligned}
\tilde\lambda_j &= \frac{2}{n+1}\frac{1}{2}\sum_{i=1}^n \frac{\sin^2(i\pi/(n+1))}{4\sin^2(j\pi/(4n))+4\sin^2(i\pi/(2n+2))}\\
&\quad+\frac{\sin^2((n+1-i)\pi/(n+1))}{4\sin^2(j\pi/(4n))+4\sin^2((n+1-i)\pi/(2n+2))}\\
&= \frac{1}{n+1}\sum_{i=1}^n \frac{\sin^2(i\pi/(n+1))(2\sin^2(j\pi/(4n))+1)}{4\sin^4(j\pi/(4n))+4\sin^2(j\pi/(4n))+\sin^2(i\pi/(n+1))}.
\end{aligned}$$

To shorten expression, we introduce two more notations

$$\xi_j = \sin^2\frac{j\pi}{4n}, \tag{3.15}$$

$$\theta_j = (1+2\xi_j)(1+3h+2\xi_j-2h\xi_j). \tag{3.16}$$

By (3.11) and (3.12), we have

$$3a_j-b_j+1 = \frac{1}{n+1}\sum_{i=1}^n \frac{\sin^2(i\pi/(n+1))\theta_j}{4\xi_j^2+4\xi_j+\sin^2(i\pi/(n+1))}. \tag{3.17}$$

We show that each term is a decreasing function of ξ_j. That is, each term

$$f_i(\xi) = \frac{(2\xi+1)(2(1+h)-(1-h)(1-2\xi))}{4\xi^2+4\xi+\sin^2(i\pi/(n+1))}$$

is a decrease function of ξ, for $\xi \in (0,1)$. By the quotient rule,

$$f_i'(\xi) = \frac{(4(1+h)+8(1-h)\xi)(4\xi^2+4\xi+\sin^2(i\pi/(n+1)))}{(4\xi^2+4\xi+\sin^2(i\pi/(n+1)))^2} - \frac{((1+3h)+4(1+h)\xi+4(1-h)\xi^2)(8\xi+4)}{(4\xi^2+4\xi+\sin^2(i\pi/(n+1)))^2}.$$

The combined numerator is

$$-\left(4(1+h)\cos^2\frac{i\pi}{n+1}+8h\right)-\left(8(1-h)\cos^2\frac{i\pi}{n+1}+16h\right)\xi-(32h)\xi^2<0.$$

As each term $f_i(\xi_j)$ is desecrating with respect to j, the sum is a desecrating function of j. We prove the lemma. □

We will find a bound for the biggest term $(3a_1-b_1)$, among all $(3a_j-b_j)$, in order to bound the c_j in (3.10). One can prove that, for all $n \ge 1$,

$$3a_1-b_1 < -7h^2/16, \tag{3.18}$$

where a_1 and b_1 are defined in (3.11) and (3.12), respectively, and $h=1/(2n)$. But our proof for (3.18) is lengthy and tedious. In this paper, we prove a worse bound only, in the next lemma.

Lemma 3.3. *If $n \ge 11$, then (cf. (3.18))*

$$3a_1-b_1 < -0.049h < -7h^2/16, \tag{3.19}$$

where a_1 and b_1 are defined in (3.11) and (3.12), respectively.

Proof. By (3.17), with the notations defined in (3.11), (3.12) and (3.13),

$$3a_1-b_1+1 = (1+3h+\frac{1-h}{2}\lambda_1^{(2n-1)})\tilde{\lambda}_1. \tag{3.20}$$

We estimate an upper bound for

$$\begin{aligned}\tilde{\lambda}_1 &= \frac{2}{n+1}\sum_{i=1}^{n}\cos^2\frac{i\pi}{2(n+1)} - \frac{2}{n+1}\sum_{i=1}^{n}\frac{\cos^2\frac{i\pi}{2(n+1)}\sin^2\frac{\pi}{4n}}{\sin^2\frac{i\pi}{2(n+1)}+\sin^2\frac{\pi}{4n}}\\ &= \frac{n}{n+1} - \frac{2\sin^2\frac{\pi}{4n}}{n+1}\sum_{i=1}^{n}\left(\frac{1+\sin^2\frac{\pi}{4n}}{\sin^2\frac{i\pi}{2(n+1)}+\sin^2\frac{\pi}{4n}}-1\right)\\ &= 1-\frac{1-2n\sin^2\frac{\pi}{4n}}{n+1} - \frac{2(1+\sin^2\frac{\pi}{4n})}{n+1}\sum_{i=1}^{n}\frac{\sin^2\frac{\pi}{4n}}{\sin^2\frac{i\pi}{2(n+1)}+\sin^2\frac{\pi}{4n}}.\end{aligned} \tag{3.21}$$

As $(\sin x/x)$ is a decreasing function of x on $(0, \pi/2)$, we have

$$\sum_{i=1}^{n} \frac{\sin^2 \frac{\pi}{4n}}{\sin^2 \frac{i\pi}{2(n+1)} + \sin^2 \frac{\pi}{4n}} > \sum_{i=1}^{n} \frac{1}{\left(\frac{i\pi}{2(n+1)} \Big/ \frac{\pi}{4n}\right)^2 + 1} > \sum_{i=1}^{n} \frac{1}{1+4i^2}$$
$$\geq \sum_{i=1}^{11} \frac{1}{1+4i^2} > 0.33462.$$

Substituting the estimate into the expression of $\tilde{\lambda}_1$,

$$\tilde{\lambda}_1 < 1 - \frac{1 + 2(1 + 2\sin^2 \frac{\pi}{4n}) \cdot 0.33462 - 2n \sin^2 \frac{\pi}{4n}}{n+1} < 1 - \frac{1.55726}{n+1}.$$

By (3.20), if $n \geq 11$,

$$3a_1 - b_1 + 1 < (1 + \frac{3}{2n} + \frac{\pi^2}{8n^2})(1 - \frac{1.55726}{n+1})$$
$$< 1 - 0.049h \leq 1 - 0.98h^2 < 1 - 7h^2/16.$$

We proved the lemma. $\square$

With the explicit eigenvalues of the reduction matrix and their bounds, we can easily choose a set of parameters γ_1, γ_2 and θ, to get a constant rate of reduction, independent of mesh size h.

Theorem 3.1. *Let $\gamma_1 = 1$, $\gamma_2 = 64h^{-1}$ and $\theta = 3/7$ in Definition 1.1. The error reduction factor (for the P_1 finite element on uniform grids shown in Figure 4) is bounded by $1/7$, independent of the grid size h,*

$$\|e_{g_1}^{m+1}\|_{L^2(\Gamma)} \leq \frac{1}{7} \|e_{g_1}^{m}\|_{L^2(\Gamma)}.$$

Proof. We will apply Lemma 2.2. By (3.11) and (3.3), $a_j > 0$. By (3.14), (3.18) and (3.12),

$$3a_j - b_j \leq 3a_1 - b_1 \leq -7h^2/16,$$
$$b_j \geq 3a_j + 7h^2/16 > 0. \tag{3.22}$$

By (3.10),

$$c_j = \frac{1 - b_j/a_j}{1 + b_j/a_j} \cdot \frac{64h^{-1} - b_j/a_j}{64h^{-1} + b_j/a_j}.$$

We let $z = b_j/a_j > 0$ in Lemma 2.2. The critical point is (may be outside the b_j/a_j range)

$$z_0 = \sqrt{\gamma_1 \gamma_2} = 8h^{-1/2}.$$

We find the two end points of possible z. First, by (3.13),

$$\tilde{\lambda}_j \geq \frac{2}{n+1} \sum_{i=1}^{n} \frac{1}{8} \sin^2 \frac{in\pi}{n+1} = \frac{n}{8(n+1)} > \frac{1}{8}.$$

Thus, by (3.11), (3.12) and (3.3),

$$
\begin{aligned}
a_j &\le (h - \frac{h}{6}\cdot 0)\cdot 1 = h,\\
a_j &\ge (h - \frac{h}{6}\cdot 4)\cdot \frac{1}{8} = \frac{h}{12},\\
b_j &\le 1-(1+\frac{1}{2}\cdot 0)\cdot\frac{1}{8} = \frac{7}{8}.
\end{aligned}
$$

In the first inequality, we used (3.21) that $\tilde{\lambda}_j < 1$. We find one end point for z:

$$\frac{b_j}{a_j} \le \frac{7/8}{h/12} = \frac{21}{2h} \equiv z_r.$$

For the other end point, by (3.22),

$$\frac{b_j}{a_j} \ge 3 + \frac{7h^2/16}{a_j} \ge 3 + \frac{7h^2/16}{h} = 3 + \frac{7h}{16} \equiv z_l.$$

By Lemma 2.2, the range of c_j is between its values at $z = z_l, z_0, z_r$. We note that $z_l < z_0 < z_r$ here. At each point, we need to apply 2.2 again for h varying. But we can find some rough (but good enough) bounds at each point, directly.

$$
\begin{aligned}
&\text{At } z = z_r: && -0.718... = -\frac{107}{149} < c_j \le -\frac{1070}{1639} = -0.65...\\
&\text{At } z = z_l: && -0.50098... = -\frac{2184975}{4361329} \le c_j < -\frac{1}{2} = -0.5.\\
&\text{At } z = z_0: && -1 < c_j \le \frac{32-129\sqrt{2}}{32-129\sqrt{2}} = -0.7015...
\end{aligned}
$$

Hence the value of c_j is always strictly between -1 and $-1/2$. When $\theta = 3/7$, we get,

$$\theta + (1-\theta)c_j > \frac{3}{7} + \frac{4}{7}(-1) = -\frac{1}{7}, \tag{3.23}$$

$$\theta + (1-\theta)c_j < \frac{3}{7} + \frac{4}{7}(-\frac{1}{2}) = \frac{1}{7}. \tag{3.24}$$

This gives the error reduction factor. □

By (3.23) and (3.24), we can get the following result for a general relaxation parameter θ.

Corollary 3.1. *Let $\gamma_1 = 1$ and $\gamma_2 = 64h^{-1}$ in Definition 1.1. The error reduction factor ρ for the P_1 finite element on uniform grids (shown in Figure 4) is*

$$\rho = \begin{cases} 1-2\theta, & 0 \le \theta \le 3/7,\\ (3\theta-1)/2, & 3/7 < \theta \le 1.\end{cases}$$

That is, $\|e_{g_1}^{m+1}\|_{L^2(\Gamma)} \le \rho \|e_{g_1}^{m}\|_{L^2(\Gamma)}$.

4. Convergence on general grids

In this section, we consider the convergence behavior of the Robin-Robin DD method on general quasi-uniform meshes. By the algorithm in Definition 1.1, for $i = 1, 2$,

$$a_i(e_i^m, v) + \gamma_i \langle e_i^m, v\rangle = \langle \varepsilon_i^m, v\rangle, \quad \forall v \in V_i$$

where the errors are defined by

$$\varepsilon_i^m = g_i - g_i^m, \quad e_1^m = u - u^m, \quad \text{and} \quad e_2^m = w - w^m.$$

Let S_1 and S_2 be the standard Dirichlet-to-Neumann operators, cf. [29, 30]. The error ε_i^n $(i = 1, 2)$, restricted to the interface Γ, satisfies the relation

$$\varepsilon_i^m = (\gamma_i + S_i) e_i^m|_\Gamma. \tag{4.1}$$

Using the first interface update (1.5), we have

$$\varepsilon_2^m = -\varepsilon_1^m + (\gamma_1 + \gamma_2) e_1^m|_\Gamma. \tag{4.2}$$

For the second one, by (1.7) and (1.8),

$$\begin{aligned}
\varepsilon_1^{m+1} &= \theta\varepsilon_1^m + (1-\theta)[-\varepsilon_2^m + (\gamma_1 + \gamma_2) e_2^m|_\Gamma] \\
&= \theta\varepsilon_1^m + (1-\theta)[-(\gamma_2 + S_2) e_2^m|_\Gamma + (\gamma_1 + \gamma_2) e_2^m|_\Gamma] \\
&= \theta\varepsilon_1^m + (1-\theta)(\gamma_1 - S_2) e_2^m|_\Gamma.
\end{aligned}$$

By (4.1), (4.2), we have

$$\begin{aligned}
\varepsilon_1^{m+1} &= \theta\varepsilon_1^m + (1-\theta)(\gamma_1 - S_2)(\gamma_2 + S_2)^{-1}\varepsilon_2^m \\
&= \theta\varepsilon_1^m + (1-\theta)(\gamma_1 - S_2)(\gamma_2 + S_2)^{-1}(\gamma_2 - S_1) e_1^m|_\Gamma \\
&= [\theta + (1-\theta)(\gamma_1 - S_2)(\gamma_2 + S_2)^{-1}(\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}]\varepsilon_1^m.
\end{aligned}$$

Let us represent the iteration by

$$\varepsilon_1^{m+1} = R\,\varepsilon_1^m,$$

where

$$R = \theta - (1-\theta) T, \tag{4.3}$$

$$T = (S_2 - \gamma_1)(\gamma_2 + S_2)^{-1}(\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}. \tag{4.4}$$

Next, we give an convergence analysis for this DD operator R.

4.1. Symmetric case: $S_1 = S_2 (:= S)$.

Let z be an eigenvector of the symmetric operator S (cf. [29, 30]) corresponding to the eigenvalue λ_s. By (4.4), z is also an eigenvector of the symmetric operator T.

$$\begin{aligned}
[\theta + (1-\theta)T]z &= [\theta + (1-\theta)\frac{(\gamma_1 - \lambda_s)(\gamma_2 - \lambda_s)}{(\gamma_1 + \lambda_s)(\gamma_2 + \lambda_s)}]z \\
&= [\theta - (1-\theta)\omega(\lambda_s)]z.
\end{aligned}$$

It is known [30] that

$$\lambda_s \in [c_0, C_0 h^{-1}].$$

Now if we choose

$$0 < \gamma_1 \le c_0, \quad \text{and} \quad \gamma_2 \ge C_0 h^{-1}, \tag{4.5}$$

by Lemma 2.2, we get

$$0 \le \omega(\lambda_s) \le \frac{(\eta-1)^2}{(\eta+1)^2}, \quad \eta = \sqrt{\frac{\gamma_2}{\gamma_1}}.$$

Then we bound the spectrum of the symmetric operator R, $\sigma(R)$, as

$$\sigma(R) \subset [\theta - (1-\theta)\frac{(\eta-1)^2}{(\eta+1)^2}, \theta] \subset [-\frac{1}{3}, \frac{1}{3}],$$

when choosing the parameter $\theta = 1/3$, cf. Remark 2.1. That is, the convergence rate is bounded by $1/3$, independent of the mesh size h, when choosing parameters by (4.5).

4.2. **Nonsymmetric case: $S_1 \approx S_2$.** In this case, there exist two positive constant $0 < s \le 1$ and $t \ge 1$, independent of the grid size h, such that for all $v \in V_i|_\Gamma$ (cf. [29] for details):

$$s(S_1 v, v) \le (S_2 v, v) \le t(S_1 v, v). \tag{A1}$$

$S_i (i = 1, 2)$ are symmetric and positive definite(SPD). Let $\underline{\lambda}_i$ be the minimum eigenvalue, and $\overline{\lambda}_i$ the maximum eigenvalue of S_i. In this subsection, we assume that the parameters are chosen to satisfy

$$0 < \gamma_1 \le \min\{\underline{\lambda}_1,\ \underline{\lambda}_2\}, \quad \text{and} \quad \gamma_2 \ge 3\max\{\overline{\lambda}_1,\ \overline{\lambda}_2\}. \tag{A2}$$

The parameter selection is similar to that in the symmetric case, (4.5).

Lemma 4.1. *The condition* (A1) *has another version*

$$\frac{1}{t}(S_1^{-1} v, v) \le (S_2^{-1} v, v) \le \frac{1}{s}(S_1^{-1} v, v). \tag{4.6}$$

Proof. Replacing v by $S_1^{-\frac{1}{2}} v$ in (A1),

$$s(v, v) \le (S_1^{-\frac{1}{2}} S_2 S_1^{-\frac{1}{2}} v, v) \le t(v, v).$$

This inequality implies that the spectrum of the SPD operator $S_1^{-\frac{1}{2}} S_2 S_1^{-\frac{1}{2}}$ is within $[s, t]$. So the spectrum of its inverse, $S_1^{\frac{1}{2}} S_2^{-1} S_1^{\frac{1}{2}}$ is inside $[t^{-1}, s^{-1}]$, i.e.,

$$\frac{1}{t}(v, v) \le (S_1^{\frac{1}{2}} S_2^{-1} S_1^{\frac{1}{2}} v, v) \le \frac{1}{s}(v, v).$$

(4.6) follows after replacing v by $S^{-\frac{1}{2}} v$. □

To find the spectrum of DD operator T in (4.4), we introduce a symmetric operator

$$\tilde{T} = (\gamma_1 + S_1)^{-\frac{1}{2}} (\gamma_2 - S_1)^{\frac{1}{2}} (S_2 - \gamma_1)(\gamma_2 + S_2)^{-1} (\gamma_2 - S_1)^{\frac{1}{2}} (\gamma_1 + S_1)^{-\frac{1}{2}}. \tag{4.7}$$

This operator is similar to the nonsymmetric operator T, defined in (4.4).

Lemma 4.2. *If Assumption* (A2) *is satisfied, then $\tilde{T}$ is SPD.*

Proof. $\tilde{T}$ is symmetric because

$$\begin{aligned}\tilde{T}^T &= (\gamma_1 + S_1^T)^{-\frac{1}{2}}(\gamma_2 - S_1^T)^{\frac{1}{2}}(\gamma_2 + S_2^T)^{-1}(S_2^T - \gamma_1)(\gamma_2 - S_1^T)^{\frac{1}{2}}(\gamma_1 + S_1^T)^{-\frac{1}{2}} \\ &= \tilde{T}.\end{aligned}$$

Notice that $(S_2-\gamma_1)(\gamma_2+S_2)^{-1} = I-(\gamma_1+\gamma_2)(\gamma_2+S_2)^{-1}$. Its minimum eigenvalue is

$$1 - (\gamma_1 + \gamma_2)(\gamma_2 + \underline{\lambda}_2)^{-1} = \frac{\underline{\lambda}_2 - \gamma_1}{\gamma_2 + \underline{\lambda}_2},$$

which is positive by Assumption (A2). Similarly, the minimum eigenvalue of $(\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}$ is $(\gamma_2 - \overline{\lambda}_1)/(\gamma_1 + \overline{\lambda}_1)$, which is also positive by Assumption (A2). Now for any $v \in V_i|_\Gamma$, we have, denoting $\tilde{v} = (\gamma_2 - S_1)^{\frac{1}{2}}(\gamma_1 + S_1)^{-\frac{1}{2}}v$,

$$\begin{aligned}(\tilde{T}v, v) &= ((S_2 - \gamma_1)(\gamma_2 + S_2)^{-1}\tilde{v}, \tilde{v}) \\ &\geq \frac{\underline{\lambda}_2 - \gamma_1}{\gamma_2 + \underline{\lambda}_2}(\tilde{v}, \tilde{v}) \\ &= \frac{\underline{\lambda}_2 - \gamma_1}{\gamma_2 + \underline{\lambda}_2}((\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}v, v) \\ &= \frac{\underline{\lambda}_2 - \gamma_1}{\gamma_2 + \underline{\lambda}_2} \cdot \frac{\gamma_2 - \overline{\lambda}_1}{\gamma_1 + \overline{\lambda}_1}(v, v).\end{aligned}$$

It means that the minimum eigenvalue of $\tilde{T}$ is greater than $\dfrac{\underline{\lambda}_2 - \gamma_1}{\gamma_2 + \underline{\lambda}_2}\dfrac{\gamma_2 - \overline{\lambda}_1}{\gamma_1 + \overline{\lambda}_1} > 0$. That is to say, the symmetric operator $\tilde{T}$ is also positive definite. □

We find an upper bound of the spectrum of SPD operator $\tilde{T}$ next. To this end, we rewrite $\tilde{T}$ as

$$\tilde{T} = \tilde{T}_2^T\tilde{T}_1\tilde{T}_2, \tag{4.8}$$

where

$$\tilde{T}_1 = (S_2 - \gamma_1)(\gamma_2 + S_2)^{-1}(\gamma_2 - S_2)(\gamma_1 + S_2)^{-1}, \tag{4.9}$$

$$\tilde{T}_2 = (\gamma_1 + S_2)^{\frac{1}{2}}(\gamma_2 - S_2)^{-\frac{1}{2}}(\gamma_1 + S_1)^{-\frac{1}{2}}(\gamma_2 - S_1)^{\frac{1}{2}}. \tag{4.10}$$

Lemma 4.3. *If* (A1) *and* (A2) *hold, then, for the t defined in* (A1)*,*

$$((\gamma_2 - S_2)^{-1}(\gamma_1 + S_2)v, v) \leq (2t - 1)\,((\gamma_2 - S_1)^{-1}(\gamma_1 + S_1)v, v). \tag{4.11}$$

Proof. By (A1) and (4.6),

$$\begin{aligned}((\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}v, v) &= ((\gamma_1 + \gamma_2)(\gamma_1 + S_1)^{-1}v, v) - (v, v) \\ &\leq t\,((\gamma_1 + \gamma_2)(\gamma_1 + S_2)^{-1}v, v) - (v, v) \\ &= t\,((\gamma_2 - S_2)(\gamma_1 + S_2)^{-1}v, v) + (t - 1)(v, v).\end{aligned}$$

We bound the second term next. By the assumption (A2),

$$((\gamma_2 - S_2)(\gamma_1 + S_2)^{-1}v, v) \geq \frac{\gamma_2 - \overline{\lambda}_2}{\gamma_1 + \overline{\lambda}_2}(v, v) \geq \frac{2\overline{\lambda}_2}{2\overline{\lambda}_2}(v, v) = (v, v).$$

Combining above two inequalities,

$$((\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}v, v) \le (2t-1)\,((\gamma_2 - S_2)(\gamma_1 + S_2)^{-1}v, v). \tag{4.12}$$

Applying Lemma 4.1, replacing S_2 there by $(\gamma_2 - S_1)(\gamma_1 + S_1)^{-1}$ and S_1 by $(\gamma_2 - S_2)(\gamma_1 + S_2)^{-1}$, by (4.12),

$$\frac{1}{2t-1}((\gamma_2 - S_2)^{-1}(\gamma_1 + S_2)v, v) \le ((\gamma_2 - S_1)^{-1}(\gamma_1 + S_1)v, v).$$

(4.11) is proved. □

Lemma 4.4. *If assumptions* (A1) *and* (A2) *hold, then the spectrum of the SPD operator* $\tilde{T}$ *is bounded by*

$$\sigma(\tilde{T}) \subset (0, 2t-1]. \tag{4.13}$$

Proof. $\tilde{T}_1$ is SPD, cf. (4.9). The eigenvalues of $\tilde{T}_1$ are

$$\tilde{\lambda}_j = \frac{\lambda_{2,j} - \gamma_1}{\gamma_2 + \lambda_{2,j}} \frac{\gamma_2 - \lambda_{2,j}}{\gamma_1 + \lambda_{2,j}},$$

where $\{\lambda_{2,j}\}$ are all eigenvalues of S_2. By (A2),

$$\tilde{\lambda}_j > \frac{\lambda_{2,j} - \underline{\lambda}_2}{\gamma_2 + \lambda_{2,j}} \frac{4\overline{\lambda}_2 - \lambda_{2,j}}{\gamma_1 + \lambda_{2,j}} \ge 0,$$

$$\tilde{\lambda}_j < \frac{\lambda_{2,j}}{\gamma_1 + \lambda_{2,j}} \frac{\overline{\lambda}_2}{\gamma_2 + \lambda_{2,j}} < 1.$$

Then, by (4.10), (4.8) and (4.11),

$$\begin{aligned} 0 < (\tilde{T}v, v) < (\tilde{T}_2 v, \tilde{T}_2 v) &= ((\gamma_1 + S_2)(\gamma_2 - S_2)^{-1}\tilde{v}, \tilde{v}) \\ &\le (2t-1)((\gamma_1 + S_1)(\gamma_2 - S_1)^{-1}\tilde{v}, \tilde{v}) = (2t-1)(v, v), \end{aligned}$$

where

$$\tilde{v} = (\gamma_1 + S_1)^{-\frac{1}{2}}(\gamma_2 - S_1)^{\frac{1}{2}}v.$$

The proof is completed. □

Theorem 4.1. *If the assumptions* (A1) *and* (A2) *hold, then the spectrum of DD reduction operator R, defined in* (4.3)*, is bounded, independent of the grid size h:*

$$\sigma(R) \subset [-\frac{2t-1}{2t+1}, \frac{2t-1}{2t+1}], \tag{4.14}$$

when θ is selected by

$$\theta = \frac{2t-1}{2t+1}. \tag{4.15}$$

Proof. By (4.13) and (4.3),

$$\sigma(R) \subset [\theta - (1-\theta)(2t-1), \theta],$$

where t is defined in (A1), independent of h. Similar to the idea in Lemma 2.1, (4.14) follows after we choose the optimal θ by (4.15). □

5. Numerical test

For numerical test, we solve the Poison equation (1.1) on the unit square $[0,1]$. The exact solution is chosen

$$u(x,y) = 2^6(x^3 - x^4)(y - y^2).$$

We choose $x = 1/2$ as the domain decomposition interface. We use P_1 conforming finite element on uniform criss grids, shown in Figure 4.

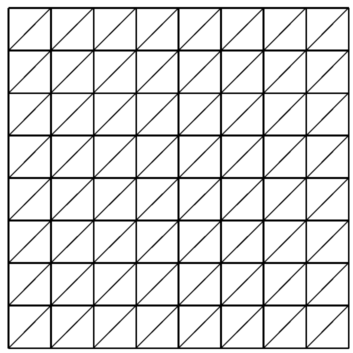

Figure 4. A uniform criss grid of size $h = 1/8$.

First, we do the Robin-Robin iteration (Definition 1.1) for problems with different grid size. The parameters used are $\gamma_1 = 1$, $\gamma_2 = 64/h$ and $\theta = 3/7$. The iteration stops when $|g_1^{m+1} - g_1^m|_{l^\infty} < 10^{-11}$. The number of iteration, the error and the order of convergence for the finite element solution are listed in Table 5.1. We note that there is a superconvergence for the finite element solution in semi-H^1 norm.

Table 5.1. The errors and the iteration numbers, by Definition 1.1.

h	$\|u_I - u_h\|_{L^2}$	h^n	$\|u_I - u_h\|_{H^1}$	h^n	#DD
1/4	0.0027120		0.203663		14
1/12	0.0000716	1.65	0.004456	1.74	14
1/20	0.0000098	1.93	0.000605	1.95	14
1/28	0.0000026	1.97	0.000159	1.98	14
1/36	0.0000009	1.99	0.000058	1.99	14
1/44	0.0000004	1.99	0.000026	1.99	14
1/52	0.0000002	1.99	0.000013	2.00	14

Next, we check our theoretic bounds in Theorem 3.1. In (3.23) and (3.24), if we vary θ from $0/7$ to $6/7$, we can get the following theoretic bounds:

$$\frac{7}{7}, \frac{5}{7}, \frac{3}{7}, \frac{1}{7}, \frac{5}{14}, \frac{8}{14}, \frac{11}{14}.$$

We compute the real bounds for these θ on various meshes, and list them in Table 5.2. We note that, when $\theta = 0/7 = 0$, the method is reduced to the traditional

Robin-Robin DD method (by other researchers, where $\gamma_1 = \gamma_2$), which converges at a rate of $1 - C\sqrt{h}$, cf. [30]. This can be seen in the first column of Table 5.2.

TABLE 5.2. The reduction rate with different θ in Definition 1.1.

$h \setminus \theta$	0	1/7	2/7	3/7	4/7	5/7	6/7
1/4	0.764	0.512	0.260	0.096	0.322	0.548	0.774
1/12	0.865	0.598	0.332	0.115	0.336	0.557	0.779
1/20	0.894	0.624	0.353	0.116	0.337	0.558	0.779
1/28	0.910	0.637	0.364	0.116	0.337	0.558	0.779
1/36	0.920	0.646	0.371	0.116	0.337	0.558	0.779
1/44	0.927	0.652	0.377	0.116	0.337	0.558	0.779
1/72	0.943	0.665	0.388	0.116	0.337	0.558	0.779
1/288	0.971	0.689	0.408	0.126	0.337	0.558	0.779
1/1152	0.985	0.702	0.418	0.134	0.337	0.558	0.779
Corollary 3.1	1.000	0.714	0.428	0.143	0.357	0.571	0.786

Finally, we compare the Robin-Robin domain decomposition method with the traditional Dirichlet-Neumann domain decomposition method. We code directly the Dirichlet-Neumann domain decomposition method, defined as follows.

Definition 5.1. *(The Dirichlet-Neumann domain decomposition method.) Given $w^0 (= 0)$ on Γ, find $u^m \in V_1$, $u^m|_\Gamma = w^m$:*

$$a_1(u^m, v) = (f, v)_{\Omega_1} \quad \forall v \in V_1 \cap H_0^1(\Omega_1).$$

Find $\tilde{w}^{m+1} \in V_2$:

$$a_2(\tilde{w}^{m+1}, v) = (f, v)_{\Omega_2} - a_1(u^m, v) \quad \forall v \in V_2,$$

where v is extended into Ω_1 with 0 nodal values. Then

$$w^{m+1} = \theta w^m + (1-\theta)\tilde{w}^{m+1}.$$

In Table 3, we list the number of Dirichlet-Neumann domain decomposition iterations for the above test problem, for various θ. It seems that no matter how to choose θ, the Dirichlet-Neumann domain decomposition method (21 iterations) is worse than the new Robin-Robin domain decomposition method (14 iterations).

TABLE 5.3. The iteration number for Dirichlet-Neumann DD (Definition 5.1.)

$h \setminus \theta$	0	0.25	0.35	0.4	0.45	0.5	0.55	0.75
1/4	88	24	22	25	29	33	38	78
1/12	237	34	21	23	26	30	35	71
1/20	392	37	22	22	25	29	33	68
1/28	548	38	23	21	24	28	32	66
1/36	705	39	23	21	24	27	31	64

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